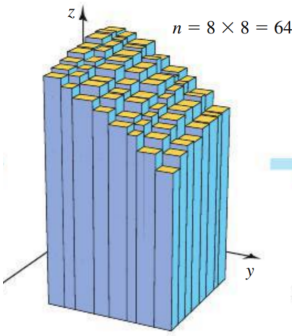
Double Integrals

To estimate the signed volume of the space between a function $z = f(x, y)$ and the xy-plane over a given region R, we can divide R into rectangles of area $\Delta A_i = \Delta x_i \Delta y_i$ and sample from each rectangle a point (x_i^*, y_i^*). The volume is then estimated by the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta A_i.$$



Definition: Double Integrals
A function f defined on a region R in the xy-plane is **integrable** if $\lim_{\Delta \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta A_i$ exists for all partitions of R and all choices of (x_i^*, y_i^*). The limit is the **double integral** of f over R

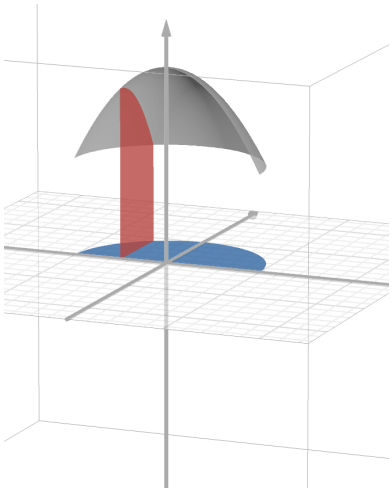
$$\iint_R f(x, y) dA = \lim_{\Delta \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta A_i.$$

This limit is rarely navigable; we instead considering the slicing method of volume calculation. Assume the y values of R lie in $[c, d]$. For fixed y^*, assume the region R consists of the points $\{(x, y) : y = y^*, a \leq x \leq b\}$. The area of the slice $y = y^*$ is $\int_g^h f(x, y^*) dx$ (note g and h depend on y^*). If f and g are continuous, then thickening slices we approximate the volume

$$V = \sum_{i=1}^{n} \int_{g(y^*)}^{h(y^*)} f(x, y^*) dx \Delta y,$$

hence

$$V = \iint_R f(x, y) dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y) dx \, dy.$$



$$f(x,y) = 2 - x^2 - y^2; R = \{(x,y) : x^2 + y^2 \leq 1, x \geq 0\}$$

Theorem 16.2: Double Integrals
Let R be a region bounded below and above by graphs of the continuous functions $y = g(x)$ and $y = h(x)$ and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint_R f(x,y)dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)dy\,dx.$$

Let R be a region bounded on the left and right by graphs of the continuous functions $x = g(y)$ and $x = h(y)$ and by the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint_R f(x,y)dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)dx\,dy.$$

Example 16.1.1. Compute $\displaystyle\iint_R \frac{x}{(1+xy)^2}dA; R = \{(x,y) : 0 \leq x \leq 4, 1 \leq y \leq 2\}$

Example 16.1.2. Suppose $f(x, y) = g(x)h(y)$ where g and h are continuous over all real numbers. Show that

$$\int_a^b \int_c^d f(x, y)dx\, dy = \left(\int_a^b g(x)dx \right) \left(\int_c^d h(y)dy \right)$$

Example 16.1.3. Compute $\iint_R y^2 dA$; R is bounded by $x = 1, y = 2x+2, y = -x-1$.

Example 16.1.4. Compute $\int_0^{\pi} \int_x^{\pi} \sin y^2 dy\, dx$

Example 16.1.5. Find the area of the region in the first quadrant bounded by $y = e^x$ and $x = \ln 2$.

Example 16.1.6. Find the volume of the solid found in the first octant bounded by the planes $x = 0, y = 0, z = 1, z = 2 - y$ and the cylinder $y = 1 - x^2$.